A relation between replica symmetry and a performance of approximation algorithms in minimum vertex cover problems

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Recently, statistical-mechanical methods have been applied to information theoretical problems such as K-satisfiable problems and constraint-satisfaction problems. These problems are deeply related to computational complexity theory, one of the main topics in theoretical computer science. Among many types of constraint-satisfaction problems, the minimum vertex cover problem (min-VC) belongs to a class of NP-hard problems. The min-VCs have been studied by the replica method [1] originally developed in the field of random spin systems.

In this presentation, we will introduce min-VCs on random α-uniform hypergraphs. An α-uniform hypergraph $G = (HV, HE)$ consists of N vertices $i \in HV$ and (hyper)edges $\{i_1, \cdots, i_\alpha\} \in HE \subset HV^\alpha$. We define covered vertices as a subset $HV' \subset HV$, and covered edges as a subset of edges connecting to at least one covered vertex. The vertex cover problem on a hypergraph G is to find covered vertices HV' by which all edges are covered. We define the cover ratio of G as $|HV'|/N$. The minimum vertex cover problem of G is to search covered vertices HV' whose cover ratio is minimum. To understand the characteristics of the graph ensemble, we consider random α-uniform hypergraphs with average degree c, whose edges are independently chosen from all α-tuples of vertices with probability $(\alpha-1)!c/N^{\alpha-1}$. Thus, we examine the average minimum cover ratio $x_c(c)$ on the random hypergraphs.

In order to adapt statistical-mechanical techniques, the vertex cover problems are mapped on the lattice gas model. We consider that at most one gas particle can be placed on each vertex of a graph, and a variable $\underline{\nu} = \{\nu_i\} = \{0,1\}^N$ represents the existence of a particle on the vertex $i \in HV$. To simplify latter notations, we define ν_i as 0 if a vertex i is covered, and ν_i as 1 if uncovered. A condition that an edge $\{i_1, \cdots, i_\alpha\} \in HE$ is covered means that $\nu_i = 0$ for at least one vertex connected with the edge. The vertex cover problem on a hypergraph G is translated into the model with the grand canonical partition function with chemical potential μ,

$$\Xi = \sum_{\underline{\nu}} \exp\left(\mu \sum_{i=1}^{N} \nu_i\right) \chi(\underline{\nu}), \quad \chi(\underline{\nu}) = \prod_{\{i_1,\cdots,i_\alpha\}\in HE} (1 - \nu_{i_1} \cdots \nu_{i_\alpha}),$$

where χ is an indicator function for the vertex cover condition. This function is 1 if all edges are covered, i.e. they connect to at least one covered vertex, and 0 otherwise. In order to consider min-VC on the random graphs, we calculate two types of averages, and take a thermodynamic limit. The minimum cover ratio represents as follows,

$$x_c(c) = 1 - \lim_{\mu\to\infty} \lim_{N\to\infty} \frac{1}{N} \mathsf{E} \left\langle \sum_i \nu_i \right\rangle_\mu ,$$

where $\langle\cdot\rangle_\mu$ is the grand canonical average and E is disorder average over the random graph ensemble. We calculate the disordered average of the grand potential by the replica method. We obtained x_c explicitly as a function of the average degree c under the replica symmetry ansatz. A critical value of c above which replica symmetry solution becomes unstable is also found.

We also examined min-VCs by two approximation algorithms, a leaf removal [2] and a linear programming. These algorithms are useful for finding an approximate solution in polynomial time. It has been suggested that performance of these approximation algorithms is related to phase transition in terms of statistical physics [3, 4].

The leaf removal algorithm was proposed to solve min-VC on $\alpha = 2$ graphs. It removes vertices called leaf and the edges connecting to the leaves from a graph and appropriately assigns covered vertices to removed vertices in a recursive step. If the graph is tree-like, the algorithm completely removes the graph and provides an exact estimate of the minimum cover ratio. However, for a loopy graph, a part of a graph with no leaves, called the core, remains at the end of the recursive steps, and the algorithm cannot solve the min-VC. We extend the algorithm to min-VCs on α-uniform hypergraphs and study the recursive procedure analytically. It is found that the estimate for x_c is identical to that obtained by the replica symmetric ansatz below the critical value of c and the algorithm fails above the value because of the emergence of a large core.

The min-VCs can be regarded as integer programming problems. However, it is known in general that the integer programming problems are solved in polynomial time on average if the number of constraints is sufficiently small compared with their variables. We examined this phenomenon for min-VCs by solving the linear programming problems, the relaxation problems of the integer programming problems. The phenomenon is reproduced below the critical value of c mentioned above, and the typical running time of the linear programming problems changes at the value. These results suggest that the difficulty to solve the problems changes at the critical value.

We will report these results in detail and would like to discuss the further study concerning these problems.

References

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