High-precision threshold of the toric code from spin-glass theory and graph polynomials

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2015/07/01

This is in collaboration with Prof. Jesper L. Jacobsen (ENS).
J. Phys. A: Math. Theor. 48 095001 (2015) [IOP select]
Supported by the JSPS core-to-core program,
and MEXT KAKENHI (No.15H03699)

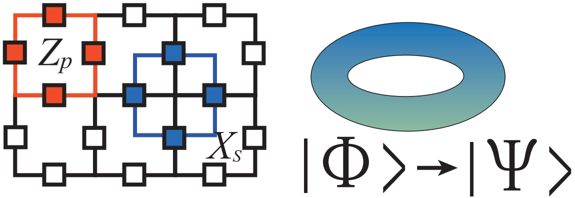
Toric code

Set a physical qubit on each edge of the square lattice on a torus.
The stabilizer operators are

$$Z_p = \prod_{(ij)\in\partial p} \sigma^z_{(ij)} \quad X_s = \prod_{(ij)\in\partial s} \sigma^x_{(ij)}.$$

They are commutable and the stabilizer state satisfies

$$Z_p|\Psi\rangle = |\Psi\rangle \ (\forall p) \quad X_s|\Psi\rangle = |\Psi\rangle \ (\forall s)$$



The block denotes the physical qubit.

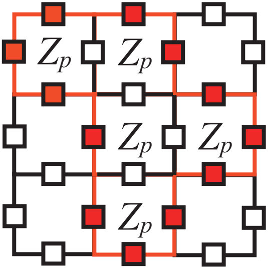
Trivial cycle = Stabilizer operators

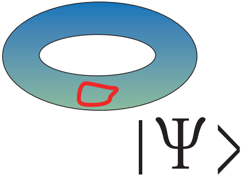
The stabilizer state can be characterized by a product of the operators

$$|\Psi(V^*, V)\rangle = \prod_{p \in V^*} Z_p \prod_{s \in V} X_s |\Phi\rangle \tag{1}$$

We use the degeneracy as redundancy of the logical qubits.

$$|\Psi_0\rangle \propto \sum_{V^*, V} |\Psi(V^*, V)\rangle$$



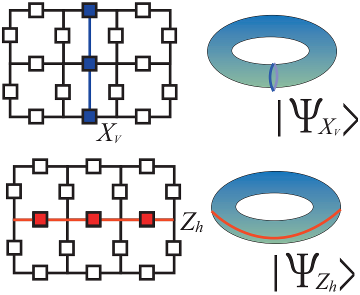


Nontrivial cycle = Logical operators

Let us introduce the "logical" operators

$$Z_h = \prod_{(ij)\in L_h} \sigma^z_{(ij)} \quad X_v = \prod_{(ij)\in L_v} \sigma^x_{(ij)},$$

and X_h and Z_v. (Z_h and X_v, which commutes with each other, Z_p and X_s)



Encode

We have four (2^2) different logical states.

$$|\Psi_{Z_h}\rangle \propto Z_h \sum_{V^*,V} |\Psi(V^*,V)\rangle.$$

Computation

We can implement the Pauli operator $\{X_h, Z_v\} = 0$ on the toric code.

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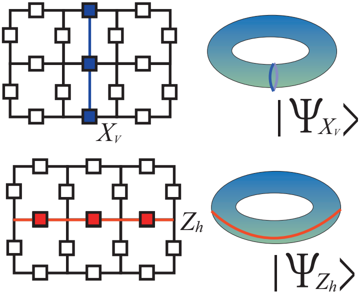

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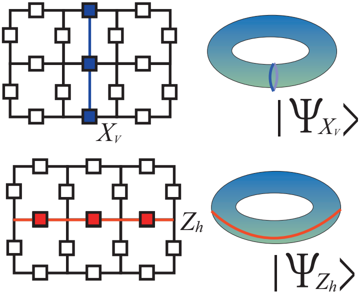

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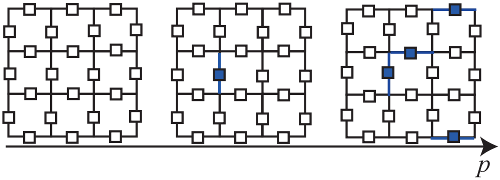
Error model

The error chain (flip ($\sigma^x_{(ij)}$) and phase ($\sigma^z_{(ij)}$) errors) appears following

$$P(E) = p^{|E|}(1-p)^{N_B-|E|} \propto \prod_{ij} \mathrm{e}^{K_p \tau^E_{ij}} \quad \left(\mathrm{e}^{2K_p} = \frac{1-p}{p}\right)$$

where $\tau^E_{ij} = 1$ for $ij \in E$ and $\tau^E_{ij} = -1$ for $ij \notin E$



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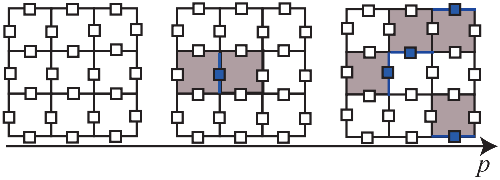

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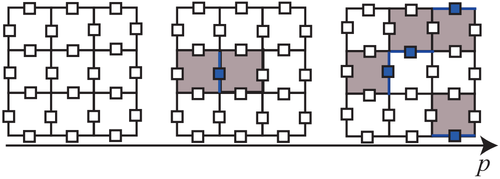

Error correction strategy

Connection between two ends of error chains

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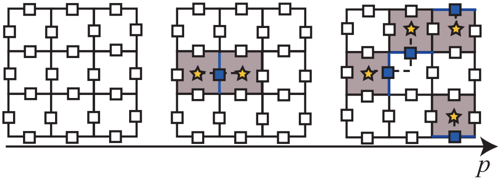

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Optimal error correction

The posterior distribution of additional chains E^* conditioned on ∂E is

$$P(E^*|\partial E) \propto \prod_{ij} \mathrm{e}^{K_p \tau_{ij}^{E^*}}$$

where $E^* + E + C = C^*$ and C^* is trivial cycle while C is nontrivial one.
The trivial cycle reads $\tau_{ij}^{E^*}\tau_{ij}^{E}\tau_{ij}^{C} = \sigma_i\sigma_j$.

Mapping to Spin-glass theory

Summation over C^* yields probability of C conditioned on ∂E.

$$P(C|\partial E) = \sum_{E^*+E+C=C^*} P(E^*|\partial E) \propto \sum_{\{\sigma_i\}} \prod_{ij} \mathrm{e}^{K_p \tau_{ij}^{C}\tau_{ij}^{E}\sigma_i\sigma_j} = Z_C(K_p)$$

where $Z_L(K_p)$ is the partition function of the Edwards-Anderson model.

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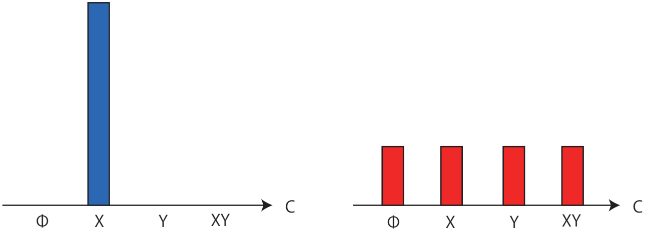
How to identify the error correctablity?

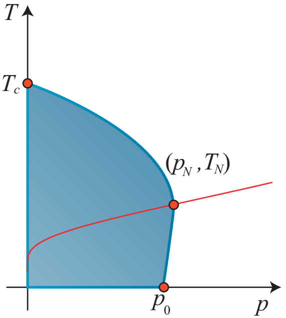
Compute the (finite but large-size) partition function with/without nontrivial cycles (Dennis 2002)

$$P(C|\partial E) = \frac{Z_C(K_p)}{Z(K_p)} = \begin{cases} 1\ (\exists C) & \text{correctable} \\ 1/4 & \text{uncorrectable} \end{cases} \qquad Z(K_p) = \sum_C Z_C(K_p)$$



My hope

Compute the precise error thresholds in analytical way!



Red: Nishimori line ($1/T = K_p$)

Possible analytical way?

Without disorder, the duality is available

$$Z(K) = \lambda^{N_B} Z(K^*)$$

where $\exp(-2K^*) = \tanh K$.
$K = K^*$ leads to the critical point.
The duality is applicable if

- Self dual (Ising, Potts models)
- Transition occurs odd times.

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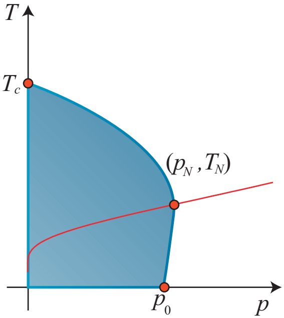

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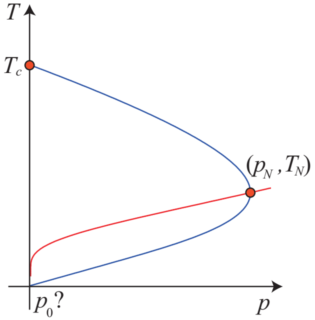
Duality for spin glass: Nishimori and Nemoto (2002)

Duality transformation with replica method estimates the location of the critical points from $[\lambda^n] = 1 \to [\log \lambda] = 0$, but it fails self-duality.

$$(1-p)\log\left(1+\mathrm{e}^{-2/T}\right) + p\log\left(1+\mathrm{e}^{2/T}\right) = \frac{1}{2}\log 2.$$

which leads to $p_N = 0.1100...$ (cf. 0.10919(7) by MCMC).



Renormalization (Ohzeki 2009)

On the renormalized system, the duality analysis leads to more precise value by $[\log \lambda_c^{(s)}] = 0$ as $p_N = 0.1092...$.

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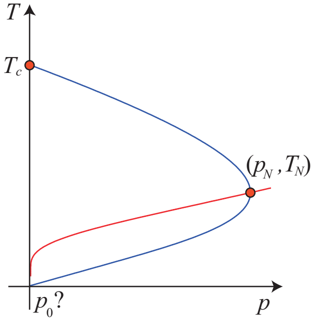

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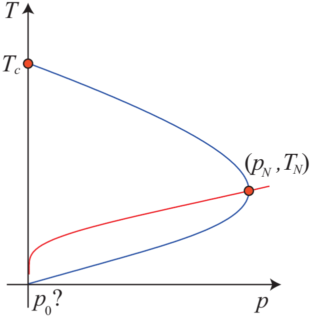

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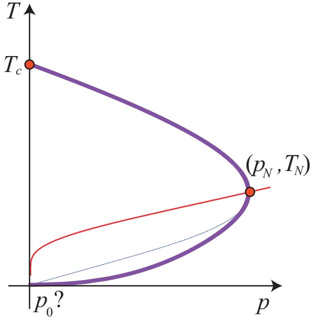

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Applications

Duality with real-space renormalization estimates error-thresholds for

- Toric code on square, triangular and hexagonal lattices [M. Ohzeki: Phys. Rev. EE 79, (2009) 021129]
- Color codes on triangular and square-octagonal lattices [M. Ohzeki: Phys. Rev. E 80 (2009) 011141]
- Toric and color codes under depolarizing channel [H. Bombin et al: Phys. Rev. X, 2 (2012) 021004]
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Why can the duality leads to precise values?

Critical polynomial

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Critical polynomial

Critical polynomial for q-state Potts model (a heuristic approach)

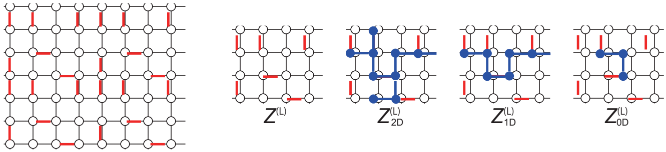
The critical points of the Potts model (generalization of the Ising model) are given by the partition function on the smallest unit

$$Z_{2\mathrm{D}}^{(L)} - qZ_{0\mathrm{D}}^{(L)} = 0,$$

where

- $Z_{2\mathrm{D}}^{(L)}$: a cluster on the torus that spans both spatial directions
- $Z_{1\mathrm{D}}^{(L)}$: a cluster that spans only one, but not both, of the directions
- $Z_{0\mathrm{D}}^{(L)}$: there are no spanning clusters.

The collection leads to the partition function as $Z^{(L)} = Z_{2\mathrm{D}}^{(L)} + Z_{1\mathrm{D}}^{(L)} + Z_{0\mathrm{D}}^{(L)}$.



Critical polynomial for Ising model: M.O. and J. L. Jacobsen (2015)

The critical polynomial can be reduced to

$$Z^{(L)} - 2Z^{(L)}_{++} = 0 \quad \left(Z^{(L)} = \sum_{\tau_x, \tau_y} Z^{(L)}_{\tau_x, \tau_z} \right)$$

This is also obtained by the duality with real-space renormalization

Critical polynomial in spin glasses

Application of the replica method yields

$$\left[\log Z^{(L)}_{++}\right] - \left[\log Z^{(L)}\right] = -\log 2.$$

and estimates $p_N = 0.10929(2)$ [Extrapolation] (cf. $p_N = 0.10919(7)$).

Interpretation in quantum error correction

- Compute the partition function

  Compute the posterior distribution

- With different boundary conditions (τ_x, τ_y)

  of different nontrivial cycles $-\tau^E_{ij}\sigma_i\sigma_j = -\tau^E_{ij}\sigma_i\sigma_j$ for $ij \in C^*$

$$\log Z_C^{(L)} - \log Z^{(L)} = \begin{cases} 0 & \text{(error correctable)} \\ -2\log 2 & \text{(error incorrectable)} \end{cases}.$$

Completely the same statements!

lack of exactness of critical polynomial

Since randomness is not periodic, there is no units.
Increase of the size of units leads to higher precision.

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Summary

We establish the analytical way to estimate the precise error thresholds from the critical polynomials.

- The similar conclusion in the quantum error correction,

$$\log Z_C^{(L)} - \log Z^{(L)} = \begin{cases} 0 & \text{correctable} \\ -2\log 2 & \text{uncorrectable} \end{cases}$$

  In our method, the critical point is determined by the middle point as

$$[\log Z_C^{(L)}] - [\log Z^{(L)}] = -\log 2.$$

- We hope a decoder of the toric code is proposed from inspiration of our method.